1. Details of Module and its structure

Module Detail	
Subject Name	Physics
Course Name	Physics 01 (Physics Part-1, Class XI)
Module Name/Title	Unit 5, Module 1, Rigid body Motion Chapter 7, System of Particles and Rotational Motion
Module Id	Keph_10701_eContent
Pre-requisites	Knowledge of kinematics, laws of motion, basic vector algebra
Objectives	After going through this lesson, the learners will be able to: • Understand the meaning of rigid body • Relate translational motion and curvilinear motion • Differentiate between rotational motion and circular motion • Understand the need to describe centre of mass • Calculate centre of mass of multiple point mass system • Know the significance of velocity and acceleration of centre of mass
Keywords	Rigid body, translatory motion, rotational motion, centre of mass

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1. UNIT SYLLABUS

Unit V: Motion of System of Particles and Rigid body

Chapter 7: System of particles and Rotational Motion

Centre of mass of a two-particle system; momentum conservation and centre of mass motion. Centre of mass of a rigid body; Centre of mass of a uniform rod.

Moment of a force; torque; angular momentum; law of conservation of angular momentum and its applications.

Equilibrium of rigid bodies; rigid body rotation and equations of rotational motion; comparison of linear and rotational motions.

Moment of inertia; radius of gyration; values of moments of inertia for simple geometrical objects (no derivation); Statement of parallel and perpendicular axes theorems and their applications.

2. MODULE-WISE DISTRIBUTION OF UNIT SYLLABUS 8 Modules

The above unit has been divided into 8 modules for better understanding.

Module 1	• Rigid body • Centre of mass • Distribution of mass • Types of motion: Translatory, circulatory and rotatory
Module 2	• Centre of mass • Application of centre of mass to describe motion • Motion of centre of mass
Module 3	• Analogy of circular motion of a point particle about a point and different points on a rigid body about an axis • Relation $v = r\omega$ • Kinematics of rotational motion
Module 4	• Moment of inertia • Difference between mass and moment of inertia • Derivation of value of moment of inertia for a lamina about a vertical axis perpendicular to the plane of the lamina • S I Unit • Radius of gyration • Perpendicular and Parallel axis theorems
Module 5	• Torque • Types of torque • Dynamics of rotational motion • $\tau = I\alpha$
Module 6	• Equilibrium of rigid bodies • Condition of net force and net torque • Applications
Module 7	• Law of conservation of angular momentum and its applications. • Applications
Module 8	• Rolling on plane surface • Horizontal • Inclined surface

	• Applications

Module 1

3. WORDS YOU MUST KNOW

- Point object: If the position of an object changes by distances much larger than the dimensions of the body the body may be treated as a point object.
- Frame of reference: Any reference frame the coordinates(x, y, z), which indicate the change in position of object with time. Inertial frame of reference is stationary or moving with a constant velocity. Non inertial frames of reference are accelerating rotating frames
- Observer: Someone who is observing objects from any frame from inertial frames such an observer is stationary with respect to the surrounding or is in uniform motion.
- Rest: A body is said to be at rest if it does not change its position with respect to its surrounding - in time.
- Motion: A body is said to be in motion if it changes its position with respect to its surroundings.

- Time elapsed: Time interval between any two observations of an object.

- Motion in one dimension: When the position of an object can be shown by change in any one coordinate out of the three (x, y, z), also called motion in a straight line.

- Motion in two dimension: When the position of an object can be shown by changes any two coordinate out of the three (x, y, z), also called motion in a plane.

- Motion in three dimension: When the position of an object can be shown by changes in all three coordinate out of the three (x, y, z).

- Distance: The path length an object has moved from its starting position to reach a final position .SI unit m, It is a scalar quantity. This can be zero or positive.

- Displacement: The distance an object has moved from its starting position moves in a particular direction. It is the shortest path length between an initial and a final position. SI unit: m. It is a vector quantity. This can be zero, positive or negative.
- Position vector: A vector representing the location of a point in space with respect to a fixed frame of reference

- Force: A push or a pull that can change the state of rest or motion of a body. It can also deform a body.

4. INTRODUCTION

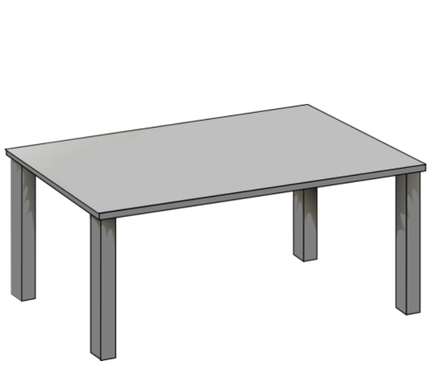
In the earlier chapters we primarily considered the motion of a single particle. (A particle is ideally represented as a point mass having no size.) We applied the results of our study even to the motion of bodies of finite size, assuming that motion of such bodies can be described in terms of the motion of a particle.
Any real body which we encounter in daily life has a finite size. In dealing with the motion of extended bodies (bodies of finite size) often the idealized model of a particle is inadequate.

In this unit we shall try to go beyond this inadequacy. We shall attempt to build an understanding of the motion of extended bodies. An extended body, in the first place, is a system of particles.

We shall begin with the consideration of motion of the system as a whole. The centre of mass of a system of particles will be a key concept here. We shall discuss the motion of the centre of mass of a system of particles and usefulness of this concept in understanding the motion of extended bodies.

Can we consider objects to be of point sized?

A point object is one that does not have spatial extension. While a non-point sized object have some spatial extension. All objects around us fall in the second category.





However such objects can be considered to be comprising of infinitesimal particles, which can be considered to be point sized.

So any object or body, which is system of large number of point size particles, is said to be rigid if it does not undergo any change in its shape or size with time, during motion or under the influence of external force.

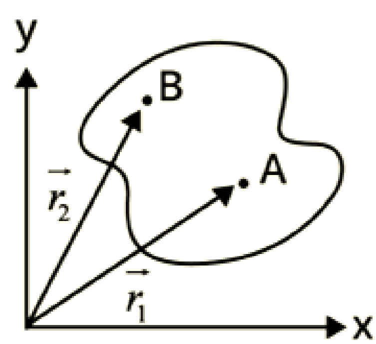
This definition, we have extended from our earlier case in Unit 2 while studying kinematics, specifically motion in one and two dimensions

One can also say the relative positions of the particles of the body remain same under the influence of external forces.

Let us consider two points A and B in the body with position vectors $\mathbf{r_1}$ and, at any time t. The body is said to be rigid if separation between A and B remains unchanged with time.

One can also say the position of point A with respect to point B will also remain unchanged with time.

$\overrightarrow{r_{BA}} = \overrightarrow{r_1} - \overrightarrow{r_2} = \text{constant}$



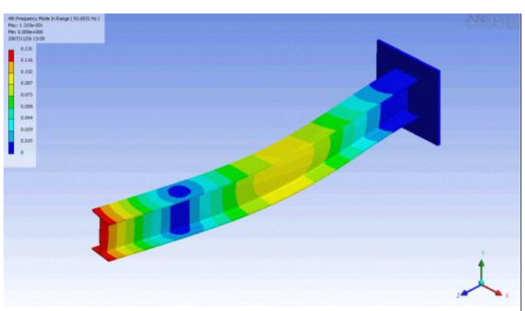
Therefore, ideally a rigid body is a body with a perfectly definite and unchanging shape. The distances between all pairs of particles of such a body do not change.

From this definition of a rigid body - no real body is truly rigid, since real bodies deform under the influence of forces. But in many situations the deformations are negligible and we can safely call them rigid. In a number of situations involving bodies such as wheels, tops, steel beams, molecules and planets, we can ignore that they warp, bend or vibrate and treat them as rigid.

Wheel with no deformation

Wheel with deformation

A seemingly rigid metal bar may bend



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We will consider the above examples to be rigid bodies.

Bodies move in different ways, we had considered motion in one dimension; it was a case of an object moving along a straight line. We called it translatory motion.

Motion in a plane was different.

We studied two options, projectile motion which clearly was in a vertical plane with constant acceleration in the downward direction and the other example we considered was motion in a circle in the horizontal and vertical planes.

Body moving in a circle maintained a fixed distance from a point (the centre of the circular path)

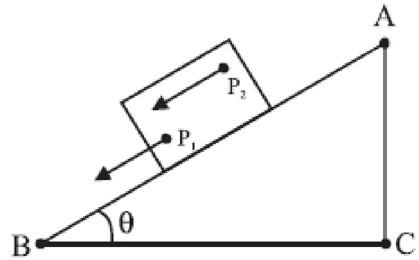
5. RIGID BODY MOTION: TRANSLATIONAL MOTION, ROTATIONAL MOTION, GENERAL PLANE MOTION AND GENERAL MOTION

What kind of motion can a rigid body have?

The motion of a rigid body can be described by extending the description of motion of point sized particles comprising the given body. **The motion of point size particles have been discussed at length in previous units of motion in a straight line and motion in a plane**.
Let us try to explore this question by taking some examples of the motion of rigid bodies.

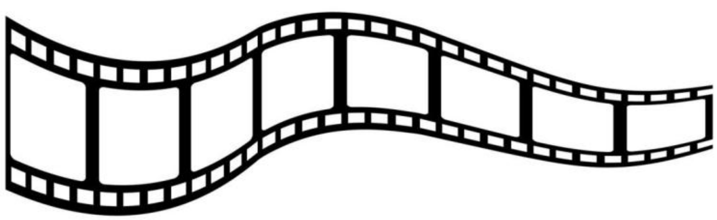
a) Translational Motion

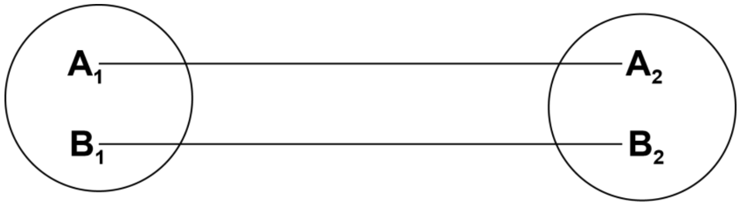
Consider a rectangular block sliding down an inclined plane without any sideways movement. The block is a rigid body. Its motion down the plane, as shown in figure below, is such that all the particles, P_1, P_2, etc of the body are moving together, i.e. they have the same velocity at any instant of time. The rigid body here is said to be in pure translational motion.

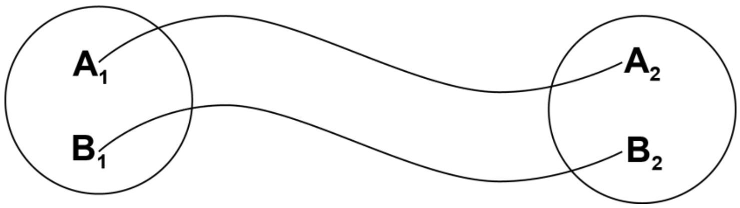


In pure translational motion at any instant of time all particles of the body have the same velocity. All the particles forming the body move along parallel paths.

If these paths are straight lines, the motion is said a rectilinear translation; if the paths are curved lines, the motion is a curvilinear motion as shown below in the figure.

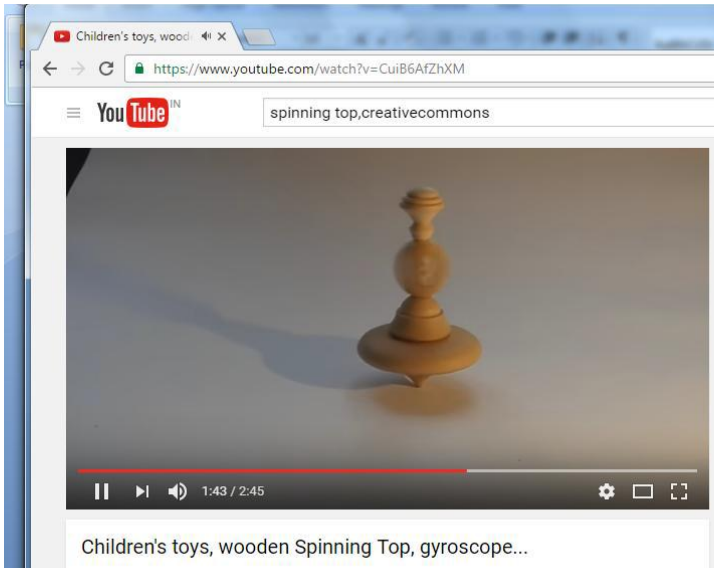






b) Rotational Motion

https://www.youtube.com/watch?v=CuiB6AfZhXM

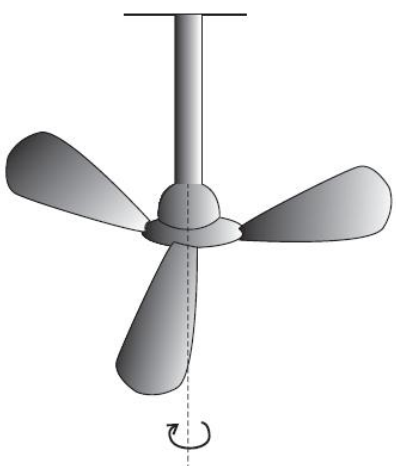


If you carefully watch a spinning top you will notice a point (**called a pivot**), or a line (**called axis**) about which the entire system moves

A rigid body is said to have rotational motion about a fixed axis when the particles forming the rigid body move in parallel planes along circles centered on the same fixed axis.

If this axis, called the axis of rotation intersects the rigid body, the particles located on the axis have zero velocity and zero acceleration.

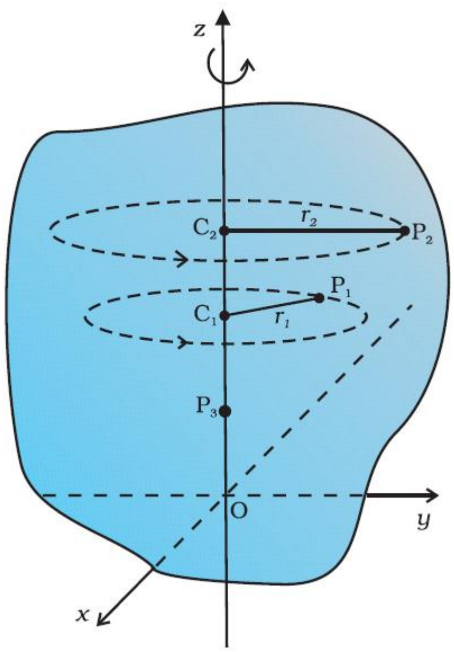
If you look around, you will come across many examples of rotation about an axis, a ceiling fan (shown in figure below), a potter's wheel, a giant wheel in a fair, a merry-go-round and so on.



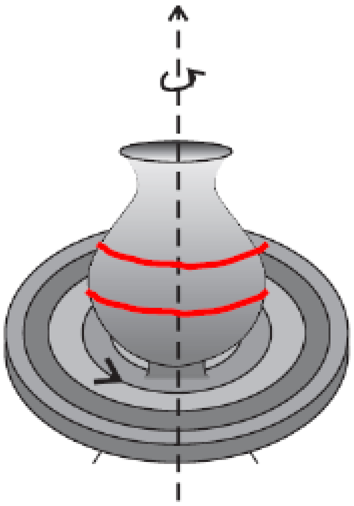
Or watch

https://www.youtube.com/watch?v=pQ-5XVEsHd4

Let us try to understand what rotation is, what characterizes rotation.



The figure above shows the rotational motion of a rigid body about a fixed axis (the z-axis of the frame of reference). For the potter's wheel as in the video this can be seen as



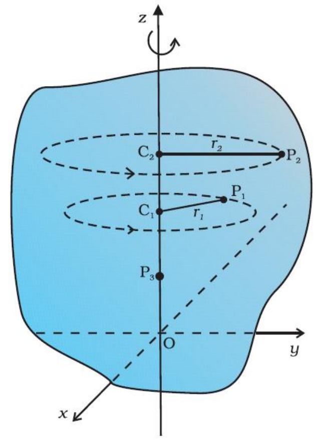
https://www.youtube.com/watch?v=lejNM8QGuP4

The potter wheel works with clay and one must not argue that the clay is not a rigid body; the idea is to imagine the circular motion of individual point masses/particles in a horizontal circle about a vertical axis.

You may notice that in rotation of a rigid body about a fixed axis, every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis.

We bring the figure here again for reference

Let P_1 be a particle of the rigid body, arbitrarily chosen and at a distance r_1 from fixed axis. The particle P_1 describes a circle of radius r_1 with its centre C_1 on the fixed axis. **The circle lies in a plane perpendicular to the axis.**



The figure also shows another particle P_2 of the rigid body, P_2 is at a distance r_2 from the fixed axis. The particle P_2 moves in a circle of radius r_2 and with centre C_2 on the axis.

This circle, too, lies in a plane perpendicular to the axis.

Note that the circles described by P_1 and P_2 may lie in different planes; both these planes, however, are perpendicular to the fixed axis.

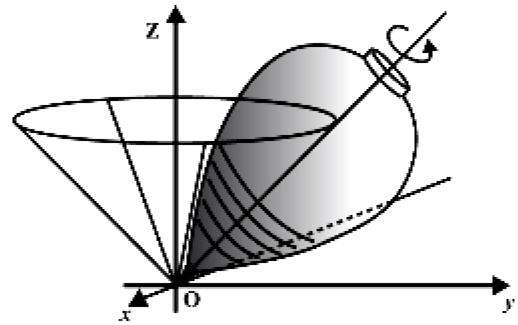
For any particle on the axis like P_3, r = 0.

All such particles, lying on the axis of rotation, remain stationary while the body rotates.

This is expected since the axis is fixed.

In some examples of rotation, however, the axis may not be fixed like the spinning top in the video [Figure given below].

Here we assume that the top does not slip from place to place and so does not have translational motion. We know from experience that the axis of such a spinning top moves around the vertical through its point of contact with the ground, sweeping out a cone as shown in figure. This movement of the axis of the top around the vertical is termed precession. Also, the point of contact, O of the top with ground is considered to be fixed here. The axis of rotation of the top at any instant passes through the point of contact, O.

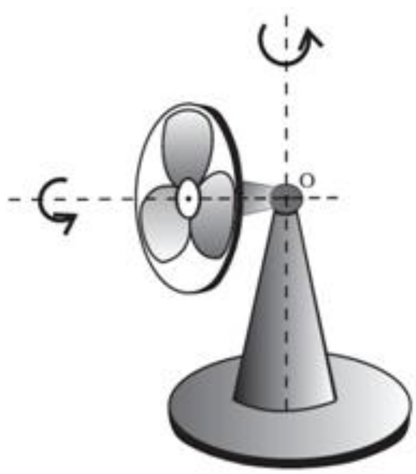


A spinning top

(The point of contact of the top with the ground. Its tip O, is fixed.)

Another simple example of this kind of rotation is the oscillating table fan or a pedestal fan.

You may have observed that the axis of rotation of such a fan has an oscillating (sidewise) movement in a horizontal plane about the vertical through the point at which the axis is pivoted (point O in Figure given below). While the fan rotates and its axis moves sidewise, the point O is fixed. Thus, in more general cases of rotation, such as the rotation of a top or a pedestal fan, one point and not one line, of the rigid body is fixed. In this case the axis is not fixed, though it always passes through the fixed point.



An oscillating table fan. The pivot of the fan, point O, is fixed.

In our study, however, we mostly deal with the simpler and special case of rotation in which one line (i.e. the axis) is fixed. Thus, for us rotation will be about a fixed axis only unless stated otherwise.

THINK ABOUT THESE

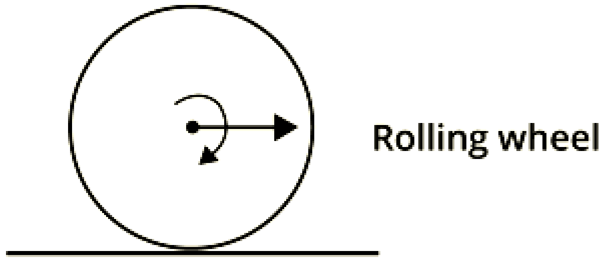
- **In pure translational motion at any instant of time all particles of the body have the same velocity.**
- **In rotation of a rigid body about a fixed axis every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis.**

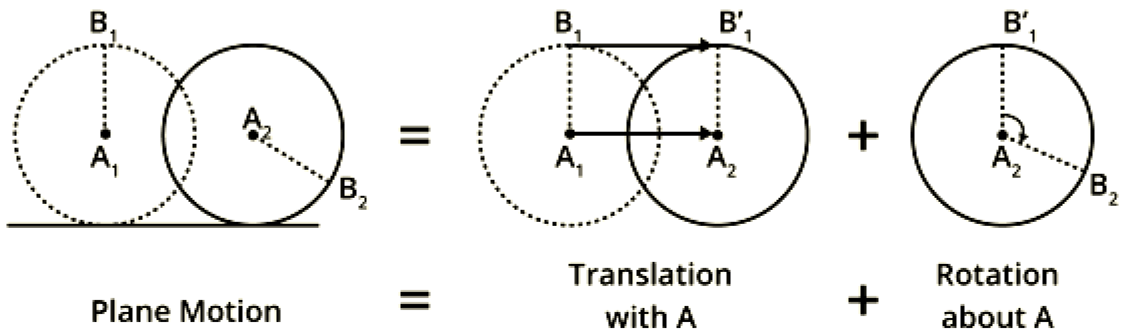
In some examples of rotation the axis may not be fixed example precession:

- **The motion of a rigid body which is not pivoted or fixed in some way is either a pure translation or a combination of translation and rotation**
- **The motion of a rigid body which is pivoted or fixed in some way is rotation**

c) GENERAL PLANE MOTION

The rolling motion of a cylinder or a wheel is a very common motion seen in daily life. If one watches different points the motion is peculiar in sense that every particle seems to be rotating and moving forward simultaneously. In fact it is a combination of rotation about a fixed axis and translation, as shown in figure.





Any plane motion which is neither a translation nor a rotation is referred as a general plane motion. Plane motion is that in which all the particles of the body move in parallel planes. Pure translational and rotational motions can also be considered to be special cases of general plane motions.

We can sum up

The most important observations which can be concluded:

- When a rigid body, in motion, is not pivoted or fixed in some way is either a pure translation or a combination of translation and rotation (Plane motion).

- The motion of a rigid body which is pivoted or fixed in some way is rotation.

The rotation may be about an axis that is fixed (e.g. a ceiling fan) or moving (e.g. revolving table fan).

We shall, in this unit, consider rotational motion about a fixed axis only.

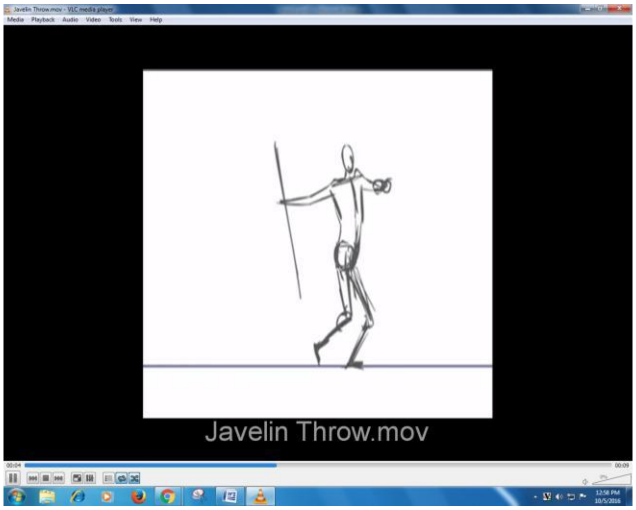
6. CENTRE OF MASS

The above discussion regarding motion of any rigid body following points came up regarding the trajectory of different particles of the body

a) In case of pure rotational motion trajectory of particles is circular

b) In case of pure translational motion trajectory of particles is linear

c) In case both translational and rotational motion the trajectory of different particles can be complicated.

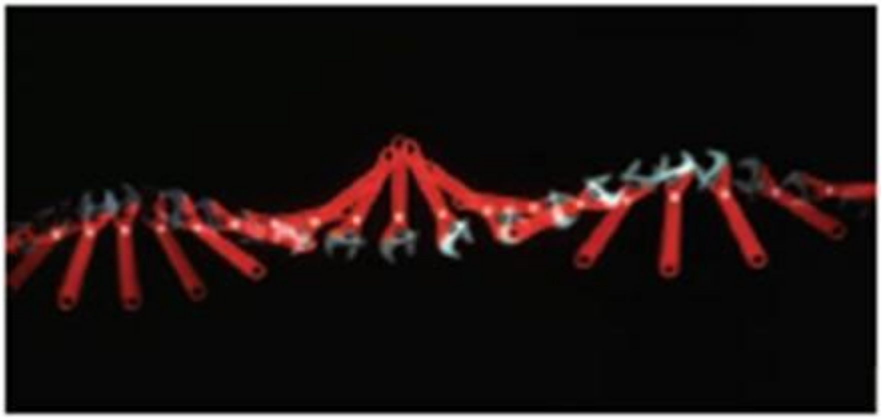


Consider a Javelin throw by an athlete. Try to visualize the trajectory of its end point. The following animation might help in visualizing the trajectory of the end point of the rod.

We've seen in projectiles the path of an object under the influence of Earth's gravitational force, it's a parabola. But that was when the object was point sized. The path traced by a point on the tip of the javelin shown in the animation is certainly not a parabola. In fact, this javelin is rotating as it travels.

Now let's do the same exercise for its centre point of rod. The path traced by a centre point is a parabola. So there is at least one point on the extended rigid body which depicts its trajectory as if it is point sized.

Now consider the following picture, taken over various instants of time, of a wrench or spanner thrown certain velocity over a horizontal smooth surface.

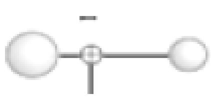


https://www.youtube.com/watch?v=DY3LYQv22qY

If one observes the point depicted as white dot, it moves in straight line. The motion of any other point is not a straight line. This special point talked about in two examples, centre of rod and white dot of the spanner can be used to describe the overall motion of the body and are defined as centre of mass of the given body.

Can such a point for a system be identified mathematically?

Let's try to do it for a simple system comprising of two point masses (different values) separated by connected by a light, rigid rod.



CASE 1

If a single force is applied at a point on the rod, closer to lighter mass (shown by small sphere in the figure), the system will move forward and simultaneously rotates clockwise. (fig. a)

CASE 2

If single force is applied at a point on the rod, very close to heavier mass (shown by bigger sphere in the figure), the system will move forward and simultaneously rotates anticlockwise.

One can explore by applying force at different points of the rod, then at a particular point, called centre of mass and denoted as CM in the figure given below. (fig. b)

CASE 3

If a single force is applied at the center of mass, the system moves in the direction of the force without rotating or it shows pure translational motion. (fig. c)

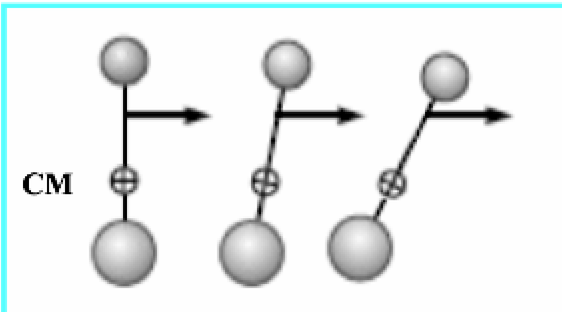


fig..(a)

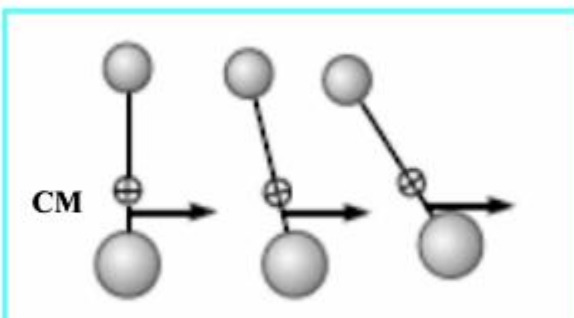


fig..(b)

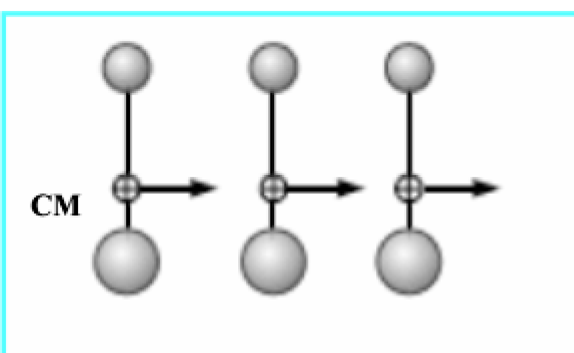


fig. (c)

The center of mass of the system, therefore for the given system is located somewhere on the line joining the two particles and is closer to the particle having the larger mass.

Hence one can define:

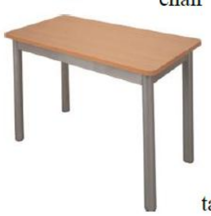
Centre of mass for system of particles as that point where if an external force acts the system shows pure translational motion.

So we can say center of mass is a useful concept.

Do you think these objects will have a center of mass?


chair


cheetah



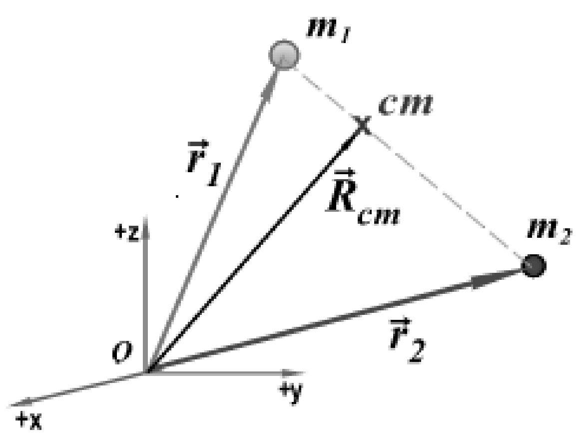
table


bus


basketball

If yes, where would it be located?

7. CENTRE OF MASS OF A TWO POINT PARTICLE SYSTEM



Consider a system of two particles of masses m_1 and m_2. Let $\vec{r_1}$ and $\vec{r_2}$ be their position vectors at any time t. Let net force acting on particle m_1 be $\vec{F_1}$ then

$$\vec{F_1} = \vec{F}_{12} + \vec{F}_{1ext}$$

Where $\vec{F}_{12}$ is force on m_1 due to m_2 and $\vec{F}_{1ext}$ is the external force on m_1.

Similarly, net force acting on particle m_2 be $\vec{F_2}$, then

$$\vec{F_2} = \vec{F}_{21} + \vec{F}_{2ext}$$

Where $\vec{F}_{21}$ is force on m_2 due to m_1 and $\vec{F}_{2ext}$ is the external force on m_2.

Using Newton's second law

$$m_1\,\vec{a_1} = \vec{F_1} = \vec{F}_{12} + \vec{F}_{1ext} \qquad \text{.........} \ (1)$$

And $$m_2\,\vec{a_2} = \vec{F_2} = \vec{F}_{21} + \vec{F}_{2ext} \qquad \text{.........} \ (2)$$

Adding above two equations (1) and (2) one gets

$$m_1\,\vec{a_1} + m_2\,\vec{a_2} = \vec{F}_{12} + \vec{F}_{1ext} + \vec{F}_{21} + \vec{F}_{2ext}$$

As $\overrightarrow{F_{21}} = -\overrightarrow{F_{12}}$ due to third law, the internal forces cancel out.

Therefore

$$m_1\,\mathbf{\vec{a_1}} + \mathbf{m_2}\,\mathbf{\vec{a_2}} = \overrightarrow{\mathbf{F_{ext}}} \qquad \text{.........} \ (3) \qquad \left(\text{As } \overrightarrow{\mathbf{F_{1ext}}} + \overrightarrow{\mathbf{F_{2ext}}} = \overrightarrow{\mathbf{F_{ext}}}\right)$$

By definition of centre of mass as that point where if an external force acts the system shows pure translational motion. Hence it can be taken as the representative for the given system for describing translational motion.

Therefore $\overrightarrow{\mathbf{F}_{ext}} = (\mathbf{m_1} + \mathbf{m_2})\,\overrightarrow{\mathbf{a}_{CM}}$ ....(4)

Using equation (3) and (4) one can write

$$\overrightarrow{\mathbf{F}_{ext}} = (\mathbf{m_1} + \mathbf{m_2})\,\overrightarrow{\mathbf{a}_{CM}} = \mathbf{m_1}\,\overrightarrow{\mathbf{a_1}} + \mathbf{m_2}\,\overrightarrow{\mathbf{a_2}}$$

The acceleration of the centre of mass of the system can be written as

$$\overrightarrow{\mathbf{a}_{CM}} = \frac{\mathbf{m_1}\,\overrightarrow{\mathbf{a_1}} + \mathbf{m_2}\,\overrightarrow{\mathbf{a_2}}}{\mathbf{m_1} + \mathbf{m_2}} \text{.......(5a)}$$

$$\frac{\mathbf{d}\,\overrightarrow{\mathbf{v}_{CM}}}{\mathbf{dt}} = \frac{\mathbf{m_1}\,\frac{\mathbf{d}\,\overrightarrow{\mathbf{v_1}}}{\mathbf{dt}} + \mathbf{m_2}\,\frac{\mathbf{d}\,\overrightarrow{\mathbf{v_2}}}{\mathbf{dt}}}{\mathbf{m_1} + \mathbf{m_2}}$$

$$\frac{\mathbf{d}\,\overrightarrow{\mathbf{v}_{CM}}}{\mathbf{dt}} = \frac{\mathbf{d}}{\mathbf{dt}}\left[\frac{\mathbf{m_1}\,\overrightarrow{\mathbf{v_1}} + \mathbf{m_2}\,\overrightarrow{\mathbf{v_2}}}{\mathbf{m_1} + \mathbf{m_2}}\right]$$

Therefore one can also write the velocity of the centre of mass as

$$\overrightarrow{\mathbf{v}_{CM}} = \frac{\mathbf{m_1}\,\overrightarrow{\mathbf{v_1}} + \mathbf{m_2}\,\overrightarrow{\mathbf{v_2}}}{\mathbf{m_1} + \mathbf{m_2}} \text{ .... (5b)}$$

$$\frac{\mathbf{d}\,\overrightarrow{\mathbf{r}_{CM}}}{\mathbf{dt}} = \frac{\mathbf{m_1}\,\frac{\mathbf{d}\,\overrightarrow{\mathbf{r_1}}}{\mathbf{dt}} + \mathbf{m_2}\,\frac{\mathbf{d}\,\overrightarrow{\mathbf{r_2}}}{\mathbf{dt}}}{\mathbf{m_1} + \mathbf{m_2}}$$

$$\frac{\mathbf{d}\,\overrightarrow{\mathbf{r}_{CM}}}{\mathbf{dt}} = \frac{\mathbf{d}}{\mathbf{dt}}\left[\frac{\mathbf{m_1}\,\overrightarrow{\mathbf{r_1}} + \mathbf{m_2}\,\overrightarrow{\mathbf{r_2}}}{\mathbf{m_1} + \mathbf{m_2}}\right]$$

Hence, the location or position vector of the centre of mass can be given as

$$\overrightarrow{r_{CM}} = \frac{m_1\,\overrightarrow{r_1} + m_2\,\overrightarrow{r_2}}{m_1 + m_2} \qquad \ldots\ldots\ (5c)$$

If the position vector of the point particles are written in terms of the position coordinates then one can deduce the expression for the position coordinates of the centre of mass of the system as

Let $\overrightarrow{r_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\overrightarrow{r_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

Let the position vector for the centre of mass be

$$\overrightarrow{r_{CM}} = x_{CM}\,\hat{i} + y_{CM}\,\hat{j} + z_{CM}\,\hat{k}$$

Using equation (5) one can write

$$\overrightarrow{r_{CM}} = x_{CM}\,\hat{i} + y_{CM}\,\hat{j} + z_{CM}\,\hat{k} = \frac{m_1(x_1\hat{i}+y_1\hat{j}+z_1\hat{k}) + m_2(x_2\hat{i}+y_2\hat{j}+z_2\hat{k})}{m_1+m_2}$$

The position coordinates of the centre of mass can be given as

$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$y_{CM} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}$$

And

$$z_{CM} = \frac{m_1z_1 + m_2z_2}{m_1 + m_2}$$

8. CENTRE OF MASS OF N- POINT PARTICLE SYSTEM

The above expressions for the position coordinates (from equation 5(a)) of the centre of masses can be extended to n particle system.

$$x_{CM} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \ldots\ldots (6a)$$

$$y_{CM} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} \ldots\ldots (6b)$$

$$z_{CM} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i} \ldots\ldots (6c)$$

The position of centre of mass, if one carefully observes the above expressions is a weighted average in which each coordinate is weighted by the mass located at that point.

The above expressions for the velocity (from equation 5(b)) of the centre of masses can be extended to ‘n’ particle system.

$$\overrightarrow{\mathbf{v_{CM}}} = \frac{\sum_{i=1}^{n} \mathbf{m_i}\overrightarrow{\mathbf{v_i}}}{\sum_{i=1}^{n} \mathbf{m_i}} \text{..........(7)}$$

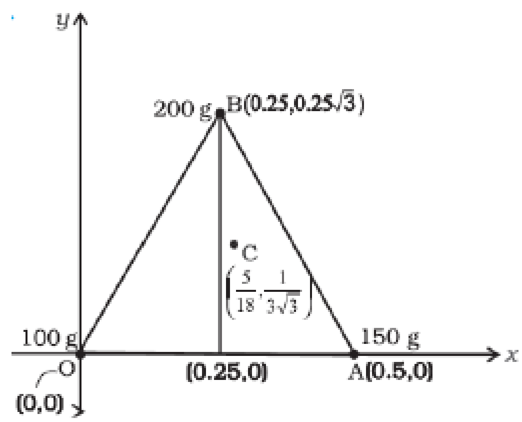
The above expressions for the acceleration (from equation 5(c)) of the centre of masses can be extended to n particle system.

$$\overrightarrow{\mathbf{a_{CM}}} = \frac{\sum_{i=1}^{n} \mathbf{m_i}\overrightarrow{\mathbf{a_i}}}{\sum_{i=1}^{n} \mathbf{m_i}} \text{..........(8)}$$

EXAMPLE

Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100g, 150g, and 200g respectively. Each side of the equilateral triangle is 0.5m long.

SOLUTION



With the x–and y–axes chosen as shown in Figure shown above. The coordinates of points O, A and B forming the equilateral triangle are respectively $(\mathbf{0,0}), (\mathbf{0.5,0}), (\mathbf{0.25, 0.25}\sqrt{\mathbf{3}})$. Let the masses 100 g, 150g and 200g be located at O, A and B be respectively. Then,

The position coordinates of the centre of mass can be given as

X coordinate for centre of mass

$$\mathbf{x_{CM}} = \frac{\mathbf{m_1x_1 + m_2x_2 + m_3x_3}}{\mathbf{m_1 + m_2 + m_3}}$$

$$\mathbf{x_{CM}} = \frac{\mathbf{100 \times 0 + 150 \times 0.5 + 200 \times 0.25}}{\mathbf{100 + 150 + 200}}$$

$$\mathbf{x_{CM}} = \frac{\mathbf{75}}{\mathbf{450}} = \frac{\mathbf{5}}{\mathbf{18}}\mathbf{m}$$

Similarly for y coordinate of centre of mass

$$\mathbf{y_{CM}} = \frac{\mathbf{m_1y_1 + m_2y_2 + m_3y_3}}{\mathbf{m_1 + m_2 + m_3}}$$

$$\mathbf{y_{CM}} = \frac{\mathbf{100 \times 0 + 150 \times 0 + 200 \times 0.25\sqrt{3}}}{\mathbf{100 + 150 + 200}}$$

$$\mathbf{y_{CM}} = \frac{\mathbf{50\sqrt{3}}}{\mathbf{450}} = \frac{\mathbf{1}}{\mathbf{3\sqrt{3}}}\mathbf{m}$$

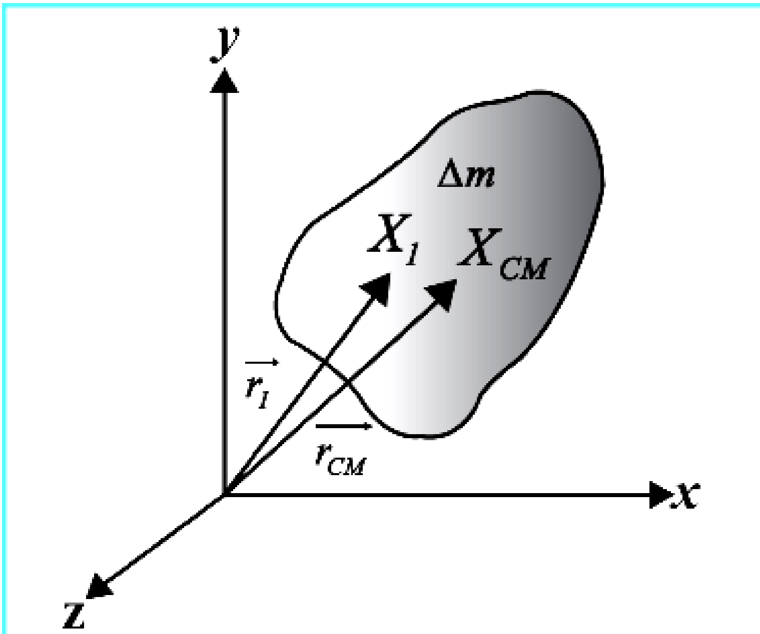
Therefore the position coordinates of the given system can be written as

$$\left(\frac{\mathbf{5}}{\mathbf{18}}\mathbf{m}, \frac{\mathbf{1}}{\mathbf{3\sqrt{3}}}\mathbf{m}\right)$$

9. CENTRE OF MASS OF SYSTEMS WITH CONTINUOUS DISTRIBUTION OF MASS

A rigid body, such as a metre ruler or a flywheel, is a system of closely packed particles; Equations 6(a), 6(b) and 6(c) therefore, applicable to a rigid body.

The number of particles (atoms or molecules) in such a body is so large that it is impossible to carry out the summations over individual particles in these equations.



Hence the need for calculus

Since the spacing of the particles is small, we can treat the body as a continuous distribution of mass.

Consider a body having continuous distribution of mass.
Let dm be the mass of an infinitesimal element, whose position coordinates are (x, y, z).
The element can be considered to be point sized.
The summation over whole body can be replaced by the integral. If the mass of the given body is M, then using equations:

$$\mathbf{x_{CM}} = \frac{\sum_{i=1}^{n} \mathbf{m_i x_i}}{\sum_{i=1}^{n} \mathbf{m_i}}$$

$$\mathbf{y_{CM}} = \frac{\sum_{i=1}^{n} \mathbf{m_i y_i}}{\sum_{i=1}^{n} \mathbf{m_i}}$$

$$z_{CM} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i}$$

One can write the expression of the coordinates for centre mass can be written as:

$$x_{cm} = \frac{1}{M}\int x\, dm \quad \text{.........(9a)}$$

$$y_{cm} = \frac{1}{M}\int y\, dm \text{.........(9b)}$$

$$z_{cm} = \frac{1}{M}\int z\, dm \text{.........(9c)}$$

Often we have to calculate the centre of mass of homogeneous bodies of regular shapes like rings, discs, spheres, rods etc.

(By a homogeneous body we mean a body with uniformly distributed mass.)

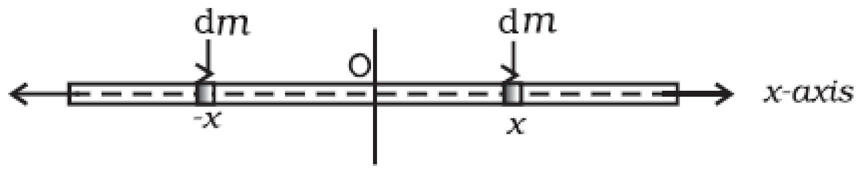
By using symmetry consideration, we can easily show that the centres of mass of these bodies lie at their geometric centres.

EXAMPLE

Find the location of the centre of mass of a thin rod of uniform distribution, using symmetry.

SOLUTION

Let us consider a thin rod, whose width and breath (in case the cross section of the rod is rectangular) or radius (in case the cross section of the rod is cylindrical) is much smaller than its length. Taking the origin to be at the geometric centre of the rod and x-axis to be along the length of the rod, we can say that on account of reflection symmetry, for every element dm of the rod at x, there is an element of the same mass dm located at –x .



The net contribution of every such pair to the integral is zero, as centre of mass each such pair is at $\frac{\mathbf{dm\,x + dm\,(-x)}}{\mathbf{dm+dm}} = \mathbf{0}$ (using 6a)

Hence the $\mathbf{x_{cm}} = \frac{\mathbf{1}}{\mathbf{M}}\int \mathbf{x\ dm}$ **is zero.**

The centre of mass of the rod is at origin, at the geometrical centre of rod. Thus, the centre of mass of a homogenous thin rod coincides with its geometric centre. This can be understood on the basis of reflection symmetry.

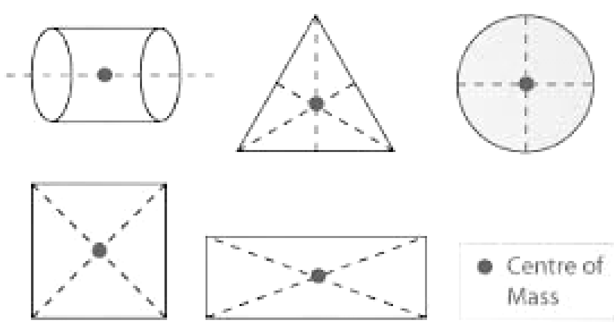
The same symmetry argument will apply to homogeneous rings, discs, spheres, or even thick rods of circular or rectangular cross section. For all such bodies one can observe that for every element dm at a point (x, y, z) one can always take an element of the same mass at the point (–x, –y, –z). (In other words, the origin is a point of reflection symmetry for these bodies.) As a result, the integrals in Eq. (9 a,b,c) all are zero.

This means that for all the above bodies, their centre of mass coincides with their geometric centre.

The following table gives the position of the centre of mass of some symmetrical bodies with uniform distribution of mass.

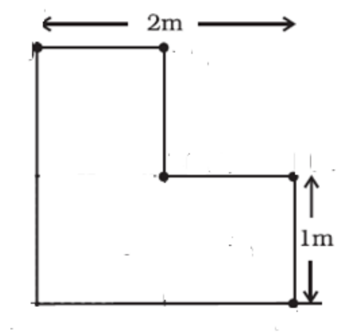
Object	Position of the centre of mass
Hollow sphere	Centre of the sphere
Solid sphere	Centre of the sphere

Circular ring	Centre of the ring
Circular disc	Centre of the disc
Rod	Centre of the rod
A plane lamina (Rectangular or square)	Point of intersection of diagonals
Triangular plane lamina	Point of intersection of medians of the triangle or centroid
Rectangular or cubical block	Point of intersection of diagonals



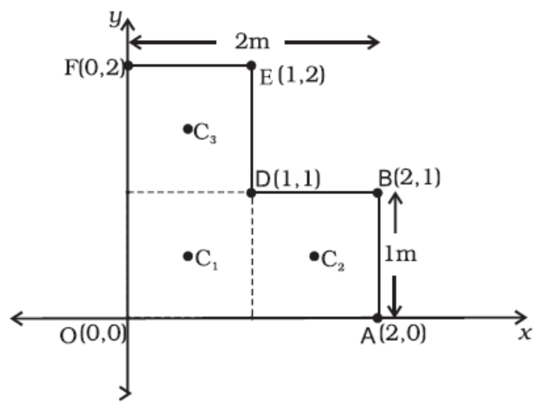
EXAMPLE

Find the centre of mass of a uniform L-shaped lamina (a thin flat plate) with dimensions as shown. The sheet is obtained by cutting one fourth part of a square shape lamina. Consider the mass of the lamina is 3 kg.



SOLUTION:

Choosing the X and Y axes the length and breadth of the lamina, as shown in Figure given below.



We have the coordinates of the vertices of the L-shaped lamina as given in the figure.

We can think of the L-shape to consist of three squares each of length 1m.

The mass of each square is 1kg, since the lamina is uniform.

The centers of mass C_1, C_2 and C_3 of the squares are, by symmetry, their geometric centres and have coordinates (1/2, 1/2), (3/2,1/2), (1/2,3/2) respectively.

We take the masses of the squares to be concentrated at these points. The centre of mass of the whole L shape (X, Y) is the centre of mass of these mass points.

Hence,

$$\mathbf{x_{CM}} = \frac{\mathbf{m_1x_1 + m_2x_2 + m_3x_3}}{\mathbf{m_1 + m_2 + m_3}}$$

$$\mathbf{x_{CM}} = \frac{\mathbf{1 \times \frac{1}{2} + 1 \times \frac{3}{2} + 1 \times \frac{1}{2}}}{\mathbf{1 + 1 + 1}} = \frac{\mathbf{5}}{\mathbf{6}}\, \boldsymbol{m}$$

Similarly,

$$\mathbf{y_{CM}} = \frac{\mathbf{m_1y_1 + m_2y_2 + m_3y_3}}{\mathbf{m_1 + m_2 + m_3}}$$

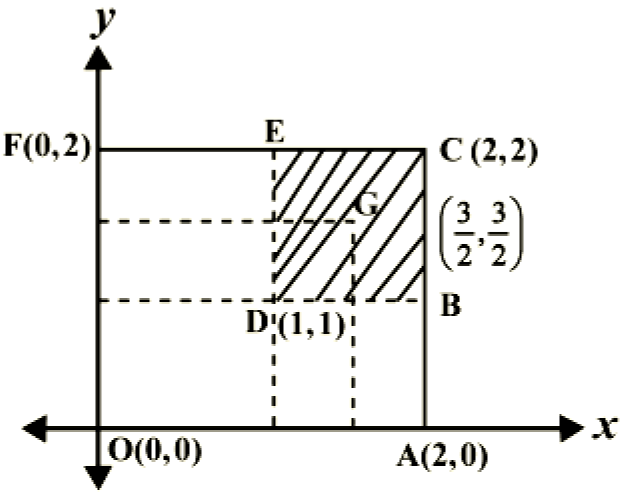
$$\mathbf{y_{CM}} = \frac{\mathbf{1 \times \frac{1}{2} + 1 \times \frac{1}{2} + 1 \times \frac{3}{2}}}{\mathbf{1 + 1 + 1}} = \frac{\mathbf{5}}{\mathbf{6}}\, \boldsymbol{m}$$

Therefore the coordinates of centre of mass are $\left(\frac{5}{6}, \frac{5}{6}\right)$ **m.**

Alternatively

The expression for position of centre of mass is a weighted average in which each coordinate is weighted by the mass located at that point; hence the above problem can be done another way. Let us consider the whole sheet or lamina be intact, its centre of mass will be at its centre i.e. point D (1, 1) m. the mass of whole sheet or lamina will be 4 kg. When we consider the whole sheet then it does have the contribution of the part removed also. If a part is removed shown as shaded part in the figure, below then its contribution has to be removed too.

The centre of mass of the removed part is at the centre of Small Square, point G, with coordinates $\left(\frac{3}{2}, \frac{3}{2}\right)$ **m.**



The expression for the centre of mass when a part of a body is removed can be written as

$$\overrightarrow{\mathbf{r}_{\mathbf{CM}}} = \frac{\mathbf{m_1}\,\overrightarrow{\mathbf{r_1}} - \mathbf{m_2}\,\overrightarrow{\mathbf{r_2}}}{\mathbf{m_1} - \mathbf{m_2}}$$

The x and y coordinates can be written as

$$\mathbf{x_{CM}} = \frac{\mathbf{m_1 x_1} - \mathbf{m_2 x_2}}{\mathbf{m_1} - \mathbf{m_2}}$$

$$\mathbf{y_{CM}} = \frac{\mathbf{m_1 y_1} - \mathbf{m_2 y_2}}{\mathbf{m_1} - \mathbf{m_2}}$$

Where,

$\mathbf{m_1}$ is the mass of whole body

$\mathbf{m_2}$ is the mass of the part of body removed

$\overrightarrow{\mathbf{r_1}}$ is position vector of the centre of mass of whole body

$\overrightarrow{\mathbf{r_2}}$ is position vector of the centre of mass of part of body removed

Here in the above problem, as shown in the figure

$\mathbf{m_1} = \mathbf{4\ kg}$ and $\mathbf{m_2} = \mathbf{2\ kg}$

Position coordinates of the centre of mass of the whole body $\mathbf{x_1 = 1m, y_1 = 1m}$

Position coordinates of the centre of mass of the part of body removed $\mathbf{x_2 = \frac{3}{2}m, y_2 = \frac{3}{2}m}$

Therefore X coordinate of the centre of mass of L shaped sheet or lamina

$$\mathbf{x_{CM} = \frac{4 \times 1 - 1 \times \frac{3}{2}}{4 - 1} = \frac{5}{6}\ m}$$

The y coordinate of the centre of mass of L shaped sheet or lamina

$$\mathbf{y_{CM} = \frac{4 \times 1 - 1 \times \frac{3}{2}}{4 - 1} = \frac{5}{6}\ m}$$

10. SUMMARY

- When a rigid body, in motion, is not pivoted or fixed in some way is either a pure translation or a combination of translation and rotation (Plane motion).
- The motion of a rigid body which is pivoted or fixed in some way is rotation.
- The rotation may be about an axis that is fixed (e.g. a ceiling fan) or moving (e.g. revolving table fan).
- Centre of mass for a system of particles or distribution of mass depends on the geometrical shape and the distribution of mass of the body.

- The position vector for the centre of mass of a two particle system is given as

$$\mathbf{\overrightarrow{r_{CM}} = \frac{m_1\,\overrightarrow{r_1} + m_2\,\overrightarrow{r_2}}{m_1 + m_2}}$$

- The expression for the position coordinates of the centre of masses for a system of n particle system.

$$\mathbf{x_{CM}} = \frac{\sum_{i=1}^{n} \mathbf{m_i x_i}}{\sum_{i=1}^{n} \mathbf{m_i}}$$

$$\mathbf{y_{CM}} = \frac{\sum_{i=1}^{n} \mathbf{m_i y_i}}{\sum_{i=1}^{n} \mathbf{m_i}}$$

$$\mathbf{z_{CM}} = \frac{\sum_{i=1}^{n} \mathbf{m_i z_i}}{\sum_{i=1}^{n} \mathbf{m_i}}$$

- For a body of continuous distribution of mass the expression of the coordinates for centre mass can be written as

$$X_{cm} = \frac{1}{M}\int x\, dm$$

$$y_{cm} = \frac{1}{M}\int y\, dm$$

$$z_{cm} = \frac{1}{M}\int z\, dm$$

- Often we have to calculate the centre of mass of homogeneous bodies of regular shapes like rings, discs, spheres, rods etc. (By a homogeneous body we mean a body with uniformly distributed mass.) By using symmetry consideration, we can easily show that the centres of mass of these bodies lie at their geometric centres.